

Adjoint Transfer and Order-Lifting Dichotomies for 2×2 Matrices over Prime-Power Rings

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11 October 2026

Abstract

For an odd prime p , we study the change in matrix order under lifting from $\mathbb{Z}/p^k\mathbb{Z}$ to $\mathbb{Z}/p^{k+1}\mathbb{Z}$. The first variation of the order defect is an affine translate of a cyclic adjoint transfer on $\mathfrak{sl}_2(\mathbb{F}_p)$. We classify the full-order transfer for every $q \in GL_2(\mathbb{F}_p)$: its ranks are 3 for scalar elements, 1 for nonscalar semisimple elements, 1 for nonscalar Jordan elements when $p = 3$, and 0 for nonscalar Jordan elements when $p > 3$. For a Jordan element $q = \lambda(I + U)$, $U^2 = 0 \neq U$, the vanishing first-order transfer for $p > 3$ admits an exact higher-order refinement. Writing $d = (p - 1)/2$, the full-order perturbation has depth d when $\text{tr}(UX) \neq 0$, depth $p - 1$ when $\text{tr}(UX) = 0$ and $X \notin \langle U \rangle$, and vanishes identically for $X \in \langle U \rangle$. For $p = 3$, among the 27 lifts of a nonscalar order-preserving parent with determinant fixed to $+1$ or -1 , exactly 9 preserve its order and 18 triple it; after the first increase, every further lift triples the order. We also give a scalar obstruction criterion for order preservation among unrestricted Jordan lifts.

2020 Mathematics Subject Classification. Primary 20H25; Secondary 20H05, 20C20.

Keywords. matrix order lifting; prime-power rings; congruence filtration; adjoint transfer; modular representation; characteristic three; Jordan perturbation; quadratic character; order defect.

1 Introduction

Let p be an odd prime. A basic lifting problem asks how the order of a matrix modulo p^k changes when the matrix is lifted to modulus p^{k+1} . The purpose of this paper is to organize this question through a tangent-space description of determinant-compatible lifts.

The relevant tangent space is $\mathfrak{sl}_2(\mathbb{F}_p)$. The first-order change of the order defect is not a bare linear invariant: it is an *affine* translate of a cyclic adjoint transfer. This distinction is essential for obtaining exact fiber counts.

The central phenomenon is especially rigid in characteristic 3. For every non-scalar reduction the relevant full-order transfer has rank one. Consequently a three-dimensional tangent space splits into affine fibers of size $3^2 = 9$, producing the exact count

$$27 = 9 + 18.$$

After the first order increase, a periodic-transfer cancellation and a prime-power propagation identity make the activated state absorbing.

The vanishing of the Jordan transfer for $p > 3$ is only a first-order statement. A higher-order analysis developed below shows that the dormant Jordan signal reappears at the exact generic perturbative depth

$$d = \frac{p - 1}{2}.$$

At that degree the response is confined to the nilpotent centralizer axis and has one-dimensional image span. For $p > 3$ this critical response is generally nonlinear as a function of X , so no linear-rank claim is intended. Its amplitude is controlled by the quadratic character of the transverse tangent coordinate. This identifies characteristic 3 not as an isolated anomaly, but as the depth-one endpoint of a uniform odd-characteristic law.

The two principal results are the global transfer classification and the exact higher-order Jordan depth law, together with the characteristic-three fixed-determinant lifting theorem. Their proofs are given below.

Relation to previous work

The congruence tangent space and its traceless Lie-algebra structure are classical [3, 4]. Modular cyclic norms and unipotent Jordan modules are standard tools in representation theory [1, 2, 8]; related adjoint-orbit problems over finite quotient rings are studied in [9]. Fibonacci periods and companion-matrix methods provide historical motivation, rather than a novelty claim [6, 7, 5]. The results below concern the specific affine order-defect formula, its exact lifting counts, and the higher-order Jordan perturbation formula. The references are not an exhaustive priority search.

2 Congruence lifts and order defects

Fix an odd prime p . Throughout we use the convention

$$\mathrm{Ad}_g(X) := gXg^{-1}.$$

Thus $\mathrm{Ad}_{q^{-j}}(X) = q^{-j}Xq^j$, consistently with the transfer sums below. Let

$$G_{p^k} = \{L \in GL_2(\mathbb{Z}/p^k\mathbb{Z}) : \det L \in \{+1, -1\}\}.$$

We consider compatible lifts from G_{p^k} to $G_{p^{k+1}}$.

Here $\det L \in \{+1, -1\}$ is an equality of residue classes in $\mathbb{Z}/p^k\mathbb{Z}$; it does not require a chosen integer representative of L to have integer determinant exactly $+1$ or -1 .

Let $q \in GL_2(\mathbb{F}_p)$ be non-scalar. Define

$$A_q = q - \frac{\mathrm{tr} q}{2}I, \quad \Delta(q) = (\mathrm{tr} q)^2 - 4\det q.$$

Cayley–Hamilton gives

$$A_q^2 = \frac{\Delta(q)}{4}I. \tag{1}$$

For an integer $r \geq 1$, define the length- r adjoint sum

$$N_q^{(r)}(X) = \sum_{j=0}^{r-1} q^{-j}Xq^j, \quad X \in \mathfrak{sl}_2(\mathbb{F}_p).$$

When $\rho(q) = \mathrm{ord}(\mathrm{Ad}_q)$, we call

$$\mathcal{N}_q = N_q^{(\rho(q))}$$

the adjoint transfer over one adjoint period.

Lemma 2.1 (Matrix order versus adjoint period). *Let*

$$r_0 = \mathrm{ord}(q), \quad \rho = \mathrm{ord}(\mathrm{Ad}_q).$$

Then $\rho \mid r_0$, q^ρ is scalar, and

$$N_q^{(r_0)} = \frac{r_0}{\rho} \mathcal{N}_q \quad \text{in } \text{End}_{\mathbb{F}_p}(\mathfrak{sl}_2(\mathbb{F}_p)). \quad (2)$$

More generally, for every multiple $r = s\rho$,

$$N_q^{(r)} = s \mathcal{N}_q.$$

Proof. The equality $\text{Ad}_q^\rho = 1$ means that q^ρ commutes with every traceless matrix. Since p is odd, every 2×2 matrix is a sum of a traceless matrix and a scalar matrix (subtract half its trace). Thus q^ρ commutes with the entire matrix algebra. Its center consists precisely of scalar matrices, so q^ρ is scalar. Since $q^{r_0} = I$, the adjoint period divides r_0 . The length- r_0 sum decomposes into r_0/ρ consecutive blocks of length ρ , and every block is the same because $\text{Ad}_q^\rho = 1$. This gives (2). $\square$

Corollary 2.2 (No period-collapse over $\mathbb{F}_p$). *Let p be prime, $q \in GL_2(\mathbb{F}_p)$, and*

$$r_0 = \text{ord}(q), \quad \rho = \text{ord}(\text{Ad}_q).$$

Then

$$s := \frac{r_0}{\rho}$$

is the multiplicative order of the scalar $q^\rho \in \mathbb{F}_p^\times I$. Consequently

$$s \mid (p-1), \quad p \nmid s,$$

and

$$N_q^{(r_0)} = s \mathcal{N}_q$$

has the same kernel, image, and rank as $\mathcal{N}_q$.

Proof. By the preceding lemma, $q^\rho = cI$ for some $c \in \mathbb{F}_p^\times$, and $\rho \mid r_0$. Write $r_0 = \rho s$. Then

$$c^s I = q^{\rho s} = q^{r_0} = I.$$

If $c^t = 1$ for some $0 < t < s$, then $q^{\rho t} = I$, contradicting the minimality of r_0 . Hence $s = \text{ord}(c)$. Since $c \in \mathbb{F}_p^\times$, its order divides $p-1$, so s is nonzero in $\mathbb{F}_p$. Equation (2) therefore differs from $\mathcal{N}_q$ only by multiplication by a nonzero field scalar, which preserves kernel, image, and rank. $\square$

Remark 2.3. *The distinction between the matrix order r_0 and the adjoint period ρ remains notationally and structurally important, but over $GL_2(\mathbb{F}_p)$ it cannot produce a rank collapse: the repetition factor r_0/ρ always lies in $\mathbb{F}_p^\times$. For $p = 3$ this specializes to $r_0/\rho \in \{1, 2\}$.*

2.1 Determinant-compatible tangent lifts

Lemma 2.4 (Tangent torsor). *Fix a determinant-compatible reference lift C_0 from modulus p^k to p^{k+1} . Every determinant-compatible lift over the same parent can be written uniquely in the form*

$$C = C_0(I + p^k X) \pmod{p^{k+1}}, \quad X \in \mathfrak{sl}_2(\mathbb{F}_p).$$

Hence there are p^3 such tangent lifts.

Proof. Modulo p^{k+1} , a relative correction is $I + p^k X$. The first-order determinant formula gives

$$\det(I + p^k X) \equiv 1 + p^k \text{tr } X \pmod{p^{k+1}}.$$

Determinant compatibility is therefore equivalent to $\text{tr } X = 0$ in $\mathbb{F}_p$. Since $\dim_{\mathbb{F}_p} \mathfrak{sl}_2 = 3$, there are p^3 possibilities. $\square$

2.2 Order defect and its first variation

Let $m = p^k$, let C be a lift, and suppose the parent order is r . Whenever

$$C^r \equiv I \pmod{m},$$

define the order defect

$$D_r(C) = \frac{C^r - I}{m} \pmod{p}.$$

Proposition 2.5 (Exact one-step order-defect dichotomy). *Let $m = p^k$, let the reduction of C modulo m have exact order r , and suppose*

$$C^r \equiv I + mD \pmod{pm}.$$

Then the order of C modulo pm is either r or pr , and

$$D = 0 \iff \text{ord}_{pm}(C) = r,$$

while

$$D \neq 0 \iff \text{ord}_{pm}(C) = pr.$$

Proof. The order modulo pm is a multiple of the exact order r modulo m . On the other hand,

$$(I + mD)^p \equiv I \pmod{pm},$$

because every nonconstant term is divisible by $pm = p^{k+1}$. Hence $C^{pr} \equiv I \pmod{pm}$, so the only possibilities are r and pr . The first occurs exactly when $C^r \equiv I \pmod{pm}$, equivalently when $D = 0$. $\square$

Proposition 2.6 (Affine first variation). *Let*

$$C = C_0(I + mX) \pmod{pm}, \quad X \in \mathfrak{sl}_2(\mathbb{F}_p),$$

and let $q = C_0 \bmod p$. Then

$$D_r(C) - D_r(C_0) = N_q^{(r)}(X). \tag{3}$$

Thus the defect family is the affine map

$$X \mapsto D_0 + N_q^{(r)}(X), \quad D_0 = D_r(C_0).$$

Proof. Expand the product

$$[C_0(I + mX)]^r$$

to first order in m . Terms containing two factors mX vanish modulo pm . Moving the single tangent insertion through successive powers of C_0 yields the conjugation sum in (3). $\square$

3 Classification of the adjoint transfer

3.1 The centralizer axis

Lemma 3.1. *For every non-scalar $q \in GL_2(\mathbb{F}_p)$,*

$$\text{Cent}(q) \cap \mathfrak{sl}_2(\mathbb{F}_p) = \langle A_q \rangle.$$

Proof. The centralizer of a non-scalar 2×2 matrix is the two-dimensional algebra $\mathbb{F}_p[I, q]$. Its traceless subspace is one-dimensional, and A_q is a nonzero traceless element of it. $\square$

Proposition 3.2 (Image-axis principle). *For every complete adjoint period,*

$$\mathrm{Im} \mathcal{N}_q \subseteq \langle A_q \rangle.$$

Consequently

$$\mathrm{rank} \mathcal{N}_q \leq 1.$$

Proof. Cyclically shifting the transfer sum shows

$$q^{-1} \mathcal{N}_q(X) q = \mathcal{N}_q(X).$$

The transfer also preserves tracelessness. Its image therefore lies in $\mathrm{Cent}(q) \cap \mathfrak{sl}_2(\mathbb{F}_p) = \langle A_q \rangle$. $\square$

3.2 Transfer classification and explicit operator formulas

We now sharpen the rank classification to an explicit classification of the full-order operator

$$N_q := N_q^{(r_0)}, \quad r_0 = \mathrm{ord}(q).$$

The earlier adjoint-period transfer $\mathcal{N}_q$ remains useful for the order-lifting argument, while the formulas below identify the full matrix-order transfer itself.

3.2.1 Nonscalar semisimple case

Assume $\Delta(q) \neq 0$, and put

$$Z_q = A_q = q - \frac{\mathrm{tr}(q)}{2} I.$$

By (1),

$$\mathrm{tr}(Z_q^2) = \frac{\Delta(q)}{2} \neq 0.$$

Lemma 3.3 (Semisimple fixed line). *If q is nonscalar semisimple, then*

$$\mathrm{Cent}(q) \cap \mathfrak{sl}_2(\mathbb{F}_p) = \langle Z_q \rangle,$$

and $p \nmid r_0$.

Proof. The centralizer statement was proved above. In the split case the eigenvalues of q lie in $\mathbb{F}_p^\times$; in the irreducible case they lie in $\mathbb{F}_{p^2}^\times$. Both multiplicative groups have order prime to p , so the order r_0 of a semisimple element is prime to p . $\square$

Theorem 3.4 (Semisimple centralizer-projection formula). *Let q be nonscalar semisimple and $r_0 = \mathrm{ord}(q)$. Define*

$$P_q(X) = \frac{\mathrm{tr}(Z_q X)}{\mathrm{tr}(Z_q^2)} Z_q.$$

Then

$$\boxed{N_q = r_0 P_q.}$$

Consequently

$$\mathrm{Im} N_q = \mathrm{Cent}(q) \cap \mathfrak{sl}_2(\mathbb{F}_p) = \langle Z_q \rangle, \quad \ker N_q = Z_q^\perp, \quad \mathrm{rank} N_q = 1.$$

Proof. Because $q^{r_0} = I$, cyclic reindexing gives

$$\mathrm{Ad}_q N_q = N_q = N_q \mathrm{Ad}_q.$$

Thus $\mathrm{Im} N_q$ lies in the fixed space of Ad_q , namely $\langle Z_q \rangle$. Since $p \nmid r_0$, the normalized average $E_q = r_0^{-1} N_q$ is the projector onto that fixed space. The trace pairing $\langle X, Y \rangle = \mathrm{tr}(XY)$ is nondegenerate on $\mathfrak{sl}_2(\mathbb{F}_p)$ for odd p , is preserved by conjugation, and $\mathrm{tr}(Z_q^2) \neq 0$. Hence the normalized cyclic average is precisely the trace-orthogonal projector onto $\langle Z_q \rangle$, which is P_q . Therefore $N_q = r_0 P_q$. $\square$

3.2.2 Modular cyclic norm viewpoint

Let $C_p = \langle g \rangle$ act on a vector space over characteristic p . Its cyclic norm element is

$$\mathbf{N} = 1 + g + \cdots + g^{p-1}.$$

Since $g^p - 1 = (g - 1)^p$, one has the classical group-algebra identity

$$\boxed{\mathbf{N} = (g - 1)^{p-1}}. \quad (4)$$

The classification of indecomposable modules for cyclic p -groups in characteristic p is classical [1, 2]. In the Jordan adjoint sector below, $(g - 1)^3 = 0$, so (4) already forces the p -step norm to vanish for $p > 3$. The explicit calculation identifies what survives at $p = 3$.

3.2.3 Nonscalar Jordan case

Assume $\Delta(q) = 0$. Put

$$\lambda = \frac{\text{tr}(q)}{2}.$$

Since q is nonscalar and invertible,

$$q = \lambda(I + U), \quad U \neq 0, \quad U^2 = 0,$$

and if $m = \text{ord}(\lambda)$, then $\text{ord}(q) = mp$. Scalar multiplication does not affect conjugation. For $j \in \mathbb{F}_p$,

$$(I + U)^j = I + jU, \quad (I + U)^{-j} = I - jU,$$

so

$$\text{Ad}_{(I+U)^j}(X) = X + j[U, X] - j^2 U X U. \quad (5)$$

Lemma 3.5 (Finite-field quadratic moment). *For odd prime p ,*

$$\sum_{j \in \mathbb{F}_p} 1 = 0, \quad \sum_{j \in \mathbb{F}_p} j = 0,$$

and

$$\sum_{j \in \mathbb{F}_p} j^2 = \begin{cases} -1, & p = 3, \\ 0, & p > 3. \end{cases}$$

Proof. The first two identities are standard. Over $\mathbb{F}_p^\times$, the sum of k -th powers vanishes unless $p - 1 \mid k$, in which case it equals -1 . For $k = 2$ and odd p , this occurs exactly at $p = 3$. $\square$

Lemma 3.6 (Rank of the exceptional channel). *For nonzero nilpotent $U \in M_2(\mathbb{F}_p)$ with $U^2 = 0$, the map*

$$J_U : \mathfrak{sl}_2(\mathbb{F}_p) \rightarrow \mathfrak{sl}_2(\mathbb{F}_p), \quad X \mapsto U X U,$$

has rank one.

Proof. After conjugation and nonzero scalar rescaling, U may be replaced by E_{12} . For

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

one obtains $E_{12} X E_{12} = c E_{12}$. Hence the image is exactly one dimensional. $\square$

Theorem 3.7 (Jordan full-order transfer dichotomy). *Let $q = \lambda(I + U)$ be nonscalar Jordan, with $m = \text{ord}(\lambda)$. Then*

$$N_q(X) = \begin{cases} mUXU, & p = 3, \\ 0, & p > 3. \end{cases}$$

Consequently

$$\text{rank } N_q = \begin{cases} 1, & p = 3, \\ 0, & p > 3. \end{cases}$$

Proof. Summing (5) over one unipotent p -cycle and using the moment lemma gives $X \mapsto UXU$ at $p = 3$ and the zero map for $p > 3$. Since $\text{ord}(q) = mp$, the full matrix-order transfer consists of m identical unipotent cycles. Hence it is $m(X \mapsto UXU)$ for $p = 3$ and zero for $p > 3$. Since $m \mid p - 1$, its coefficient is nonzero modulo p ; the rank-one statement follows from the preceding lemma. $\square$

Remark 3.8 (Why characteristic 3 is exceptional). *The Jordan conjugation orbit is quadratic in the cyclic parameter. Its constant and linear moments vanish for every odd prime. The quadratic moment vanishes for $p > 3$, but at $p = 3$ its degree equals $p - 1$, so it survives. Thus characteristic 3 is the critical odd characteristic for this 2×2 adjoint problem.*

3.2.4 Unified global transfer theorem

The scalar case is also transparent: if $q = \lambda I$ and $r_0 = \text{ord}(\lambda)$, then $\text{Ad}_q = I$ and

$$N_q = r_0 I.$$

Since $r_0 \mid p - 1$, this has rank three.

Theorem 3.9 (Global Adjoint Transfer Classification). *Let p be an odd prime and $q \in GL_2(\mathbb{F}_p)$. Put*

$$N_q = \sum_{j=0}^{\text{ord}(q)-1} \text{Ad}_{q^{-j}} \quad \text{on } \mathfrak{sl}_2(\mathbb{F}_p).$$

Then exactly one of the following occurs:

$$N_q = \begin{cases} r_0 I, & q = \lambda I, \quad r_0 = \text{ord}(\lambda), \\ r_0 P_q, & q \text{ nonscalar semisimple}, \\ m(X \mapsto UXU), & q = \lambda(I + U) \text{ Jordan}, \quad p = 3, \\ 0, & q = \lambda(I + U) \text{ Jordan}, \quad p > 3. \end{cases}$$

Accordingly,

$$\text{rank } N_q = \begin{cases} 3, & q \text{ scalar}, \\ 1, & q \text{ nonscalar semisimple}, \\ 1, & q \text{ nonscalar Jordan}, \quad p = 3, \\ 0, & q \text{ nonscalar Jordan}, \quad p > 3. \end{cases}$$

Proof. The scalar formula follows from trivial conjugation. The nonscalar semisimple formula is the centralizer-projection theorem. The repeated-root case is the Jordan full-order transfer dichotomy. These cases are mutually exclusive and exhaustive for 2×2 matrices over a field. $\square$

Corollary 3.10 (Global kernel–image dimensions). *Under Theorem 3.9,*

$$(\dim \ker N_q, \dim \operatorname{Im} N_q) = \begin{cases} (0, 3), & q \text{ scalar}, \\ (2, 1), & q \text{ nonscalar semisimple}, \\ (2, 1), & q \text{ Jordan, } p = 3, \\ (3, 0), & q \text{ Jordan, } p > 3. \end{cases}$$

Proof. The ambient space $\mathfrak{sl}_2(\mathbb{F}_p)$ has dimension three. Applying rank–nullity to the four cases of Theorem 3.9 gives the stated kernel and image dimensions. $\square$

3.2.5 Response–transfer corollary

Let $B_1, \dots, B_s \in M_2(\mathbb{F}_p)$ be trace probes whose restrictions span $\mathfrak{sl}_2(\mathbb{F}_p)^*$, and define

$$R_q(X) = (\operatorname{tr}(B_1 N_q(X)), \dots, \operatorname{tr}(B_s N_q(X))).$$

Corollary 3.11 (Response–transfer duality). *For every $q \in GL_2(\mathbb{F}_p)$,*

$$\boxed{\ker R_q = \ker N_q, \quad \operatorname{rank} R_q = \operatorname{rank} N_q.}$$

Thus the same 3/1/1/0 classification is the exact first-order observational reconstruction classification for a nondegenerate trace-probe family.

Proof. The trace pairing on $\mathfrak{sl}_2(\mathbb{F}_p)$ is nondegenerate for odd p , and the assumed probes span its dual. Hence the measurement map is injective on $\mathfrak{sl}_2(\mathbb{F}_p)$. Therefore all responses vanish exactly when $N_q(X) = 0$, giving equality of kernels and, by rank–nullity, equality of ranks. $\square$

Remark 3.12 (Dormancy versus measurement blindness). *If $N_q(X) \neq 0$ but a restricted probe family annihilates it, the failure is measurement blindness. If $N_q(X) = 0$, the first-order signal has already vanished before measurement. The nonscalar Jordan sector for odd $p > 3$ is therefore an exact transfer-dormant sector.*

4 Higher-order Jordan perturbations

The first-order classification above leaves a precise question in the nonscalar Jordan sector for $p > 3$: does the vanishing transfer represent absolute loss of perturbative information, or only delayed visibility? The following theorem gives an exact answer.

Let

$$q = \lambda(I + U), \quad U^2 = 0, \quad U \neq 0,$$

with $\lambda \in \mathbb{F}_p^\times$, and let

$$A(\varepsilon) = q(I + \varepsilon X), \quad X \in \mathfrak{sl}_2(\mathbb{F}_p).$$

Write

$$m = \operatorname{ord}_{\mathbb{F}_p^\times}(\lambda), \quad d = \frac{p-1}{2}.$$

Since $I + U$ has order p and $m \mid p-1$, the matrix q has order mp .

Lemma 4.1 (Traceless Cayley–Hamilton decomposition). *Let R be a commutative ring in which 2 is invertible, and let $A \in M_2(R)$. Put*

$$t = \operatorname{tr}(A), \quad Z = A - \frac{t}{2}I, \quad \Delta(A) = t^2 - 4 \det(A).$$

Then

$$Z^2 = \frac{\Delta(A)}{4}I.$$

If moreover R is an $\mathbb{F}_p$ -algebra with p odd, then

$$A^p = \left(\frac{t}{2}\right)^p I + \left(\frac{\Delta(A)}{4}\right)^{(p-1)/2} Z.$$

Proof. The matrix Z is traceless, so Cayley–Hamilton gives $Z^2 = -\det(Z)I$. Directly, $-\det(Z) = \Delta(A)/4$. For the second assertion, the scalar matrix $(t/2)I$ commutes with Z , and the intermediate binomial coefficients vanish in characteristic p . Hence

$$A^p = \left(\frac{t}{2}I + Z\right)^p = \left(\frac{t}{2}\right)^p I + Z^p.$$

Writing $p = 2d + 1$ and using $Z^2 = (\Delta/4)I$ gives $Z^p = (\Delta/4)^d Z$. $\square$

Lemma 4.2 (Jordan discriminant expansion). *For $A(\varepsilon) = q(I + \varepsilon X)$ as above, put $c = \text{tr}(UX)$. Then the discriminant is exactly*

$$\Delta(A(\varepsilon)) = \lambda^2(4c\varepsilon + (c^2 - 4\det X)\varepsilon^2).$$

In particular its linear term is $4\lambda^2 c\varepsilon$, and

$$A(0) - \frac{\text{tr } A(0)}{2}I = \lambda U.$$

Proof. Since U and X are traceless,

$$\text{tr } A(\varepsilon) = \lambda(2 + \varepsilon \text{tr}(UX)).$$

Moreover $\det(I + U) = 1$, while for a traceless 2×2 matrix X ,

$$\det(I + \varepsilon X) = 1 + \varepsilon^2 \det X.$$

Substitution into $\Delta = (\text{tr } A)^2 - 4\det A$ gives the displayed exact quadratic formula. At $\varepsilon = 0$, $\text{tr}(q) = 2\lambda$, so the traceless part of q is λU . $\square$

Theorem 4.3 (Jordan Perturbative Depth Theorem). *Let p be an odd prime,*

$$q = \lambda(I + U) \in GL_2(\mathbb{F}_p), \quad U^2 = 0, \quad U \neq 0,$$

and let $X \in \mathfrak{sl}_2(\mathbb{F}_p)$. Put

$$m = \text{ord}_{\mathbb{F}_p^\times}(\lambda), \quad d = \frac{p-1}{2}.$$

Then in $M_2(\mathbb{F}_p[[\varepsilon]])$,

$$[q(I + \varepsilon X)]^{mp} = I + m(\text{tr}(UX))^d U \varepsilon^d + O(\varepsilon^{d+1}).$$

Equivalently,

$$[\varepsilon^j][q(I + \varepsilon X)]^{mp} = 0 \quad (1 \leq j < d),$$

while

$$[\varepsilon^d][q(I + \varepsilon X)]^{mp} = m(\text{tr}(UX))^d U.$$

Proof. Apply Lemma 4.1 to $A = A(\varepsilon)$:

$$A^p = \left(\frac{\text{tr } A}{2}\right)^p I + \left(\frac{\Delta(A)}{4}\right)^d \left(A - \frac{\text{tr } A}{2} I\right).$$

The Frobenius map sends a power series $\sum a_j \varepsilon^j$ to $\sum a_j^p \varepsilon^{jp}$. Since $d < p$, the scalar term has no positive-degree contribution through degree d . By Lemma 4.2,

$$\left(\frac{\Delta(A)}{4}\right)^d = \lambda^{2d} (\text{tr}(UX))^d \varepsilon^d + O(\varepsilon^{d+1}).$$

Now $2d = p - 1$, so $\lambda^{2d} = 1$, and the constant term of the traceless factor is λU . Therefore

$$A^p = \lambda I + \lambda (\text{tr}(UX))^d U \varepsilon^d + O(\varepsilon^{d+1}).$$

Factoring out λ and raising to the m -th power gives

$$A^{mp} = I + m (\text{tr}(UX))^d U \varepsilon^d + O(\varepsilon^{d+1}),$$

because $\lambda^m = 1$ and every product of two positive-degree terms has degree at least $2d \geq d + 1$. $\square$

Corollary 4.4 (Generic Jordan perturbative depth). *If $\text{tr}(UX) \neq 0$, then the first nonzero full-order perturbative coefficient occurs exactly at*

$$d_{\min}(p) = \frac{p-1}{2}.$$

If $\text{tr}(UX) = 0$, Theorem 4.3 gives vanishing through degree d , but does not by itself determine the first later nonzero degree.

Proof. If $\text{tr}(UX) \neq 0$, then its d -th power is nonzero in the field. Theorem 4.3 shows that every lower positive-degree coefficient vanishes and that the coefficient at degree d is nonzero. When $\text{tr}(UX) = 0$, that same theorem only implies vanishing through degree d , as asserted. $\square$

Definition 4.5 (Full-order perturbative depth). *For the present Jordan problem define*

$$R_X(\varepsilon) = [q(I + \varepsilon X)]^{mp} - I.$$

The full-order perturbative depth is

$$D_R(X) = \min\{j \geq 1 : [\varepsilon^j] R_X(\varepsilon) \neq 0\},$$

with $D_R(X) = \infty$ if $R_X(\varepsilon) \equiv 0$ as a polynomial matrix over $\mathbb{F}_p$. Thus infinite depth below refers only to this full-order perturbative response.

Theorem 4.6 (Three-Layer Jordan Reconstruction-Depth Theorem). *Under the hypotheses of Theorem 4.3, let*

$$H_U = \{X \in \mathfrak{sl}_2(\mathbb{F}_p) : \text{tr}(UX) = 0\}.$$

Then

$$D_R(X) = \begin{cases} d, & X \notin H_U, \\ p-1, & X \in H_U \setminus \langle U \rangle, \\ \infty, & X \in \langle U \rangle, \end{cases} \quad d = \frac{p-1}{2}.$$

For the second layer the leading coefficient has the invariant form

$$\boxed{[\varepsilon^{p-1}][q(I + \varepsilon X)]^{mp} = m(-\det X)^d U, \quad X \in H_U \setminus \langle U \rangle.}$$

In fact $(-\det X)^d = 1$ on this stratum, so the coefficient is mU . On the final axis one has the exact identity

$$\boxed{[q(I + \varepsilon X)]^{mp} = I \quad (X \in \langle U \rangle).}$$

Proof. The first stratum is Corollary 4.4. It remains to analyze H_U . Simultaneous conjugation of q, U, X preserves $\text{tr}(UX)$, $\det X$, membership in $\langle U \rangle$, and the full-order response up to conjugation, so we may take

$$U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Writing a general traceless matrix as

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

one has $\text{tr}(UX) = c$. Hence on H_U ,

$$X = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \quad \det X = -a^2.$$

Moreover $X \in \langle U \rangle$ exactly when $a = 0$.

Suppose first that $X \in H_U \setminus \langle U \rangle$, so $a \neq 0$. The exact discriminant formula of Lemma 4.2 becomes

$$\Delta(A(\varepsilon)) = 4\lambda^2 a^2 \varepsilon^2.$$

Thus, with $d = (p-1)/2$,

$$\left(\frac{\Delta}{4}\right)^d = a^{p-1} \varepsilon^{p-1}.$$

Here $\lambda^{p-1} = 1$. Also $\text{tr } A = 2\lambda$, so the scalar term in Lemma 4.1 is λI . Since

$$A - \lambda I = \lambda(U + \varepsilon(X + UX)),$$

we obtain the exact polynomial identity

$$A^p = \lambda [I + a^{p-1} \varepsilon^{p-1} U + a^{p-1} \varepsilon^p (X + UX)].$$

Raising to the m -th power shows that the coefficient of ε^{p-1} in A^{mp} is $ma^{p-1}U = mU$, and no smaller positive degree occurs. Because $-\det X = a^2$, this is precisely

$$m(-\det X)^d U.$$

This proves the second stratum and its invariant coefficient formula.

Finally, if $X = bU$, then

$$q(I + \varepsilon X) = \lambda [I + (1 + b\varepsilon)U].$$

Since $U^2 = 0$, in characteristic p

$$[I + sU]^p = I + psU = I.$$

Therefore $A^p = \lambda I$ and $A^{mp} = I$ identically in ε . Hence $D_R(X) = \infty$ on the final axis. $\square$

Corollary 4.7 (Exact Jordan response flag). *In the setting of Theorem 4.6, the invariant flag*

$$\mathfrak{sl}_2(\mathbb{F}_p) \supset H_U \supset \langle U \rangle$$

has dimensions 3, 2, 1, respectively. Consequently its three response strata have cardinalities

$$|\mathfrak{sl}_2(\mathbb{F}_p) \setminus H_U| = p^2(p-1), \quad |H_U \setminus \langle U \rangle| = p(p-1), \quad |\langle U \rangle| = p.$$

The corresponding full-order perturbative depths are

$$\frac{p-1}{2}, \quad p-1, \quad \infty.$$

In particular, the second finite response depth is exactly twice the first.

Proof. The linear functional $X \mapsto \text{tr}(UX)$ is nonzero, so H_U is a codimension-one subspace of the three-dimensional space $\mathfrak{sl}_2(\mathbb{F}_p)$. Hence $|H_U| = p^2$. Since $U \neq 0$, the axis $\langle U \rangle$ is one-dimensional and has p elements. Taking successive set differences gives the displayed cardinalities, while the depths are those of Theorem 4.6. $\square$

Corollary 4.8 (Jordan discriminant-contact law). *In the setting of Theorem 4.6, with the convention $v_\varepsilon(0) = \infty$,*

$$v_\varepsilon(\Delta_X) = \begin{cases} 1, & X \notin H_U, \\ 2, & X \in H_U \setminus \langle U \rangle, \\ \infty, & X \in \langle U \rangle, \end{cases}$$

and consequently

$$D_R(X) = \frac{p-1}{2} v_\varepsilon(\Delta_X).$$

This identity is asserted only for the present 2×2 Jordan perturbation problem; no general discriminant-depth principle is claimed.

Proof. The exact expansion in Lemma 4.2 has a nonzero linear term when $\text{tr}(UX) \neq 0$. On H_U , its quadratic coefficient equals $-4\lambda^2 \det X$. In the coordinates used in Theorem 4.6, this is nonzero precisely when $X \notin \langle U \rangle$. On $\langle U \rangle$ the entire discriminant vanishes. Comparing these three valuations with the three depths of Theorem 4.6 gives the formula. $\square$

Corollary 4.9 (Critical Jordan axis and quadratic character). *At the critical degree, the response lies in*

$$\langle U \rangle = C_{\mathfrak{sl}_2(\mathbb{F}_p)}(q).$$

For $\text{tr}(UX) \neq 0$, Euler's criterion gives

$$[\varepsilon^d][q(I + \varepsilon X)]^{mp} = m \left(\frac{\text{tr}(UX)}{p} \right) U,$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol.

Proof. Theorem 4.3 places the critical coefficient in $\langle U \rangle$. For $c = \text{tr}(UX) \neq 0$, Euler's criterion in $\mathbb{F}_p$ gives $c^{(p-1)/2} = (\frac{c}{p})$. Substitute this identity into the coefficient formula of that theorem. $\square$

Corollary 4.10 (Recovery of the characteristic-three Jordan law). *For $p = 3$, one has $d = 1$, and*

$$[\varepsilon][q(I + \varepsilon X)]^{3m} = m \operatorname{tr}(UX)U.$$

For a nonzero nilpotent 2×2 matrix U and traceless X , $UXU = \operatorname{tr}(UX)U$. Hence

$$[\varepsilon][q(I + \varepsilon X)]^{3m} = mUXU,$$

which is exactly the full-order characteristic-three Jordan transfer law of Theorem 3.9.

Proof. At $p = 3$ one has $d = 1$. Conjugating U to E_{12} and writing $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ gives $UXU = cU = \operatorname{tr}(UX)U$. The first-order coefficient of Theorem 4.3 is therefore $mUXU$, as required. $\square$

Remark 4.11 (First-order dormancy as delayed visibility). *Theorem 4.3 does not alter the first-order transfer classification: for nonscalar Jordan elements, $N_q = 0$ when $p > 3$. It explains that vanishing as a depth effect. Characteristic 3 is the unique odd characteristic for which $(p - 1)/2 = 1$; for larger odd primes the same Jordan channel first becomes visible at higher perturbative order.*

5 Order lifting in characteristic three

5.1 Affine fiber geometry

Suppose the relevant length- r norm in (3) has rank one and the affine equation

$$D_0 + N_q^{(r)}(X) = 0$$

is solvable. Then every zero fiber is a coset of the kernel.

Since

$$\dim \mathfrak{sl}_2(\mathbb{F}_p) = 3, \quad \operatorname{rank} N = 1,$$

we have

$$\dim \ker N = 2, \quad |\ker N| = p^2.$$

Corollary 5.1 (Rank-one branching). *In a rank-one solvable affine state, exactly p^2 of the p^3 determinant-compatible tangent lifts have zero defect and*

$$p^3 - p^2 = p^2(p - 1)$$

have nonzero defect.

Proof. If the affine equation has a solution X_0 , its solution set is $X_0 + \ker N$. Rank-nullity gives $\dim \ker N = 2$, so there are p^2 zero-defect solutions among the p^3 tangent lifts. All remaining $p^3 - p^2$ lifts have nonzero defect. $\square$

Thus the local decomposition is

$$\boxed{p^3 = p^2 + p^2(p - 1)}.$$

5.2 Order-preserving states and persistence

Here PRE (order-preserving) means exact order $r_0 = \text{ord}(q)$; ACT (activated) means that the order has acquired a positive power of 3 relative to r_0 .

We now specialize to $p = 3$. Let $q \in GL_2(\mathbb{F}_3)$ be non-scalar and write

$$r_0 = \text{ord}(q).$$

By the transfer classification and the general no-period-collapse result, the full-order transfer $N_q^{(r_0)}$ has rank one and

$$\text{Im } N_q^{(r_0)} = \langle A_q \rangle.$$

Lemma 5.2 (Defect-axis lemma). *Let $C_k \in G_{3^k}$ have exact order r_0 , and let C_0 be any determinant-compatible lift to modulus 3^{k+1} . Write*

$$C_0^{r_0} = I + 3^k D \pmod{3^{k+1}}, \quad D \in M_2(\mathbb{F}_3).$$

Then

$$[q, D] = 0, \quad \text{tr } D = 0,$$

and hence

$$D \in \text{Cent}(q) \cap \mathfrak{sl}_2(\mathbb{F}_3) = \langle A_q \rangle.$$

Proof. Since C_0 commutes with its power $C_0^{r_0}$,

$$0 = [C_0, C_0^{r_0}] \equiv 3^k [C_0, D] \pmod{3^{k+1}}.$$

Dividing by 3^k and reducing modulo 3 gives $[q, D] = 0$.

Next, determinant compatibility gives $\det C_0 = \varepsilon$ with $\varepsilon \in \{\pm 1\}$ at modulus 3^{k+1} . Because the reduction of C_0 modulo 3^k has order r_0 , its determinant satisfies $\varepsilon^{r_0} = 1$; in particular, if $\varepsilon = -1$, then r_0 is even. Thus

$$1 = \det(C_0^{r_0}) = \det(I + 3^k D) \equiv 1 + 3^k \text{tr } D \pmod{3^{k+1}},$$

so $\text{tr } D = 0$ in $\mathbb{F}_3$. For non-scalar q , its centralizer in $M_2(\mathbb{F}_3)$ is $\langle I, q \rangle$; intersecting with the traceless subspace leaves the one-dimensional axis $\langle A_q \rangle$. $\square$

Theorem 5.3 (PRE-Persistence Theorem). *Let $q \in GL_2(\mathbb{F}_3)$ be non-scalar and let $r_0 = \text{ord}(q)$. Suppose $C_k \in G_{3^k}$ reduces to q modulo 3 and has exact order r_0 . Then among the 27 determinant-compatible lifts of C_k to modulus 3^{k+1} , exactly 9 have exact order r_0 and exactly 18 have exact order $3r_0$. In particular, every PRE state has PRE descendants at the next level, so PRE persistence holds for all $k \geq 1$.*

Proof. Choose any determinant-compatible reference lift C_0 . By the defect-axis lemma,

$$D_{r_0}(C_0) \in \langle A_q \rangle.$$

In characteristic three the full-order transfer has rank one and

$$\text{Im } N_q^{(r_0)} = \langle A_q \rangle.$$

Consequently

$$-D_{r_0}(C_0) \in \text{Im } N_q^{(r_0)},$$

so the affine correction equation

$$D_{r_0}(C_0) + N_q^{(r_0)}(X) = 0$$

is automatically solvable. The determinant-compatible lifts form an affine torsor over the three-dimensional space $\mathfrak{sl}_2(\mathbb{F}_3)$. Since $N_q^{(r_0)}$ has rank one, its kernel has dimension two and contains exactly $3^2 = 9$ elements. Hence the zero-defect affine fiber has exactly 9 elements. By the exact one-step order-defect dichotomy, these 9 lifts have exact order r_0 , while the other 18 have exact order $3r_0$.

Every one of the nine order-preserving children again has exact order r_0 , the same non-scalar reduction q , and determinant ± 1 . Therefore the same argument applies at the next level and, by induction, at every subsequent level. $\square$

Corollary 5.4 (PRE step with fixed determinant). *For every non-scalar characteristic-three PRE state,*

$$\boxed{\text{PRE} \longrightarrow 9\text{PRE} + 18\text{ACT}.}$$

Proof. This is the transition count asserted by the PRE-Persistence Theorem: its nine order-preserving children are PRE and its eighteen children of order $3r_0$ are newly activated. $\square$

5.3 Unrestricted Jordan lifts: a scalar obstruction criterion

The determinant condition in the preceding PRE-persistence theorem is essential to that theorem's automatic solvability conclusion. This section treats *all* lifts in GL_2 , without fixing the determinant at the next level. The resulting obstruction is not automatically zero. In particular, the statements below must not be confused with the 27-element determinant-compatible torsor.

Lemma 5.5 (Cyclic-basis divisibility). *Let $R = \mathbb{Z}/3^k\mathbb{Z}$ and let $A \in M_2(R)$ have nonscalar reduction q modulo 3. There is a vector $w \in R^2$ such that (w, Aw) is an R -basis. Consequently, if $uA + vI = I$ in $M_2(R)$, then $u = 0$ and $v = 1$ in R .*

Proof. A nonscalar 2×2 matrix over a field has a cyclic vector: otherwise every one-dimensional subspace would be invariant, forcing it to be scalar. Choose $\bar{w}$ with $(\bar{w}, q\bar{w})$ linearly independent over $\mathbb{F}_3$, and lift it to $w \in R^2$. The determinant of (w, Aw) is a unit in R , hence this pair is a basis. Applying $(uA + (v - 1)I)$ to w gives $uAw + (v - 1)w = 0$; the basis property forces $u = v - 1 = 0$. $\square$

Lemma 5.6 (Jordan full-matrix norm). *Let $q = \lambda I + N \in GL_2(\mathbb{F}_3)$, with $N \neq 0$, $N^2 = 0$, and $\lambda \in \mathbb{F}_3^\times$. For $n = \text{ord}(q) \in \{3, 6\}$ define*

$$\mathcal{N}_{q,n}(X) = \sum_{j=0}^{n-1} q^{-j} X q^j, \quad X \in M_2(\mathbb{F}_3).$$

Then

$$\mathcal{N}_{q,n}(X) = \lambda^{n-1} \text{tr}(NX)N.$$

In particular its image is $\langle N \rangle$, its rank on the four-dimensional full matrix space is one, and the trace map on its kernel is surjective.

Proof. Write $q = \lambda(I + U)$ with $U = \lambda^{-1}N$, so $U^2 = 0$. Then $(I + U)^j = I + jU$ and $(I + U)^{-j} = I - jU$ in characteristic three. The conjugation sum for $n = 3$ is

$$\sum_{j=0}^2 (I - jU)X(I + jU) = \sum_{j=0}^2 (X + j(XU - UX) - j^2 UXU) = UXU,$$

since $\sum_{j=0}^2 j = 0$ and $-\sum_{j=0}^2 j^2 = 1$ in $\mathbb{F}_3$. For a rank-one nilpotent U , the identity $UXU = \text{tr}(UX)U$ follows by choosing a Jordan basis. For $n = 6$, the conjugation sum

is twice the $n = 3$ sum, and the coefficient becomes λ^5 in the displayed formula ($\lambda = 1$ when $n = 3$, $\lambda = -1$ when $n = 6$). Finally $I \in \ker \mathcal{N}_{q,n}$ and $\text{tr}(I) = 2 \neq 0$ in $\mathbb{F}_3$, proving trace-surjectivity. $\square$

Theorem 5.7 (Unrestricted Jordan order-preserving lifts). *Fix $k \geq 1$, put $m = 3^k$, and let $A \in GL_2(\mathbb{Z}/m\mathbb{Z})$ reduce to a nonscalar Jordan matrix $q = \lambda I + N$ over $\mathbb{F}_3$. Assume that A has exact order $n = \text{ord}(q) \in \{3, 6\}$ (a PRE parent). Choose any integral representative A_0 of A , and set $t = \text{tr}(A_0)$, $d = \det(A_0)$. Define integral polynomials $u_j(t, d), v_j(t, d)$ by*

$$(u_0, v_0) = (0, 1), \quad (u_1, v_1) = (1, 0), \quad u_{j+1} = tu_j + v_j, \quad v_{j+1} = -du_j.$$

Thus $A_0^n = u_n A_0 + v_n I$ by Cayley–Hamilton. By Lemma 5.5, $m \mid u_n$ and $m \mid (v_n - 1)$. Define

$$\alpha = (u_n/m) \bmod 3, \quad \beta = ((v_n - 1)/m) \bmod 3, \quad s = \lambda\alpha + \beta \in \mathbb{F}_3.$$

Then the following are equivalent:

- (i) $s = 0$;
- (ii) *the order defect of A_0 lies in the image of the full first-variation operator;*
- (iii) *there exists a lift $B \in GL_2(\mathbb{Z}/3^{k+1}\mathbb{Z})$ of A with exact order n .*

If these conditions hold, there are exactly 27 order-preserving lifts among the 81 lifts of A , with exactly 9 in each of the three possible determinant lifts. If $s \neq 0$, none of the 81 lifts preserves order. The condition $s = 0$ is independent of the chosen integral representative A_0 .

Proof. Since $A_0^n \equiv I \pmod{m}$, the Cayley–Hamilton recurrence and Lemma 5.5 give the stated divisibilities. Modulo $3m$ the order defect is

$$E = \frac{A_0^n - I}{m} \bmod 3 = \alpha q + \beta I = \alpha N + sI.$$

Every lift of A can be written uniquely as $B = A_0(I + mX) \pmod{3m}$, with $X \in M_2(\mathbb{F}_3)$. First-order expansion gives

$$\frac{B^n - I}{m} \equiv E + \mathcal{N}_{q,n}(X) \pmod{3}.$$

By Lemma 5.6, the image of the norm is $\langle N \rangle$. As I and N are linearly independent, E belongs to that image if and only if $s = 0$. In that case the solution set is a coset of the norm kernel, which has $3^{4-1} = 27$ elements. The trace functional is surjective on the kernel, so each of its three trace fibers has $27/3 = 9$ solutions. Since

$$\det B \equiv \det A_0(1 + m \text{tr } X) \pmod{3m},$$

these are exactly the three determinant fibers. The one-step order dichotomy identifies zero defect with exact order n . Changing A_0 changes E by an element of the norm image, hence cannot change whether s vanishes; equivalently the scalar component of E modulo $\langle N \rangle$ is intrinsic. $\square$

Corollary 5.8 (Reconciliation with determinant-compatible persistence). *If, in addition, $\det A = \varepsilon \in \{+1, -1\}$ modulo 3^k , then the 27 lifts with determinant exactly ε modulo 3^{k+1} include exactly 9 order-preserving lifts and 18 order-increasing lifts.*

Proof. Choose a reference lift A_0 with determinant exactly ε modulo $3m$. As $n = 3$ or 6 and $\varepsilon^n = 1$, we have $\det(A_0^n) = 1 \pmod{3m}$, so $\text{tr } E = 0$. But $E = \alpha N + sI$ has trace $2s$ in $\mathbb{F}_3$. Thus $s = 0$, and the theorem gives nine solutions in the prescribed determinant fiber. The other eighteen lifts in that fiber increase the order by three. $\square$

Remark 5.9 (Why the two enumerations are different). *For the 16 nonscalar Jordan roots modulo 3, there are 432 order-preserving first-level lifts if all three determinant fibers are admitted. At modulus 9, an exhaustive audit classifies these 432 parents as 144 with $s = 0$ and 288 with $s \neq 0$; their unrestricted next-level order-preserving counts are respectively 27 and 0. This does not contradict the determinant-constrained PRE theorem: the obstructed parents have determinant outside the fixed ± 1 classes. The finite counts corroborate, but are not used in the proof of, Theorem 5.7. The observed 9-versus-18 division of the 27 unrestricted order-preserving children into next-level unobstructed and obstructed parents is a separate finite observation, not asserted here as an all-level branching theorem.*

5.4 Absorbing activation

Lemma 5.10 (Characteristic-three propagation). *Let $k \geq 1$ and let E be a matrix over the integers (or over a compatible 3-adic matrix ring). Then*

$$(I + 3^k E)^3 \equiv I + 3^{k+1} E \pmod{3^{k+2}}.$$

Proof. The binomial expansion is valid because I commutes with E :

$$(I + 3^k E)^3 = I + 3^{k+1} E + 3^{2k+1} E^2 + 3^{3k} E^3.$$

For $k \geq 1$ both $2k + 1 \geq k + 2$ and $3k \geq k + 2$. □

Lemma 5.11 (First-activation boundary and universal propagation). *Let $k \geq 1$, and let B be a matrix modulo 3^{k+1} whose reduction modulo 3^k has exact order r . Suppose that B has exact order $3r$ modulo 3^{k+1} . Then there is a uniquely determined nonzero matrix $D \in M_2(\mathbb{F}_3)$ such that*

$$B^r \equiv I + 3^k D \pmod{3^{k+1}}, \quad D \neq 0.$$

For every lift $\tilde{B}$ of B modulo 3^{k+2} , regardless of determinant restrictions, one has

$$\boxed{\tilde{B}^{3r} \equiv I + 3^{k+1} D \pmod{3^{k+2}}}.$$

In particular, every such lift has exact order $9r$ modulo 3^{k+2} . More generally, for any $n \geq 0$ and any lift $\hat{B}_n$ of B modulo 3^{k+2+n} ,

$$\boxed{\hat{B}_n^{3^{n+1}r} \equiv I + 3^{k+1+n} D \pmod{3^{k+2+n}}}.$$

Consequently its reduction modulo 3^{k+1+n} has exact order $3^{n+1}r$.

Proof. Since B reduces to a matrix of order r , the one-step order-defect dichotomy implies that $B^r \equiv I + 3^k D \pmod{3^{k+1}}$ for a unique $D \in M_2(\mathbb{F}_3)$, and that $D \neq 0$ because the order of B is $3r$. For any next lift $\tilde{B}$ there is a matrix E such that

$$\tilde{B}^r \equiv I + 3^k D + 3^{k+1} E \pmod{3^{k+2}}.$$

Apply Lemma 5.10 to the matrix $D + 3E$ to obtain

$$\tilde{B}^{3r} = (\tilde{B}^r)^3 \equiv I + 3^{k+1}(D + 3E) \equiv I + 3^{k+1} D \pmod{3^{k+2}}.$$

The right-hand defect is nonzero, hence the exact one-step dichotomy, applied to the parent order $3r$, forces exact order $9r$. At each successive level the same argument applies with k increased by one and the current order multiplied by three. The leading matrix $D \pmod{3}$ is unchanged. Induction proves the asserted formula and exact orders. No enumeration, rank assumption, or determinant condition is used in this propagation step. □

Proposition 5.12 (Absorbing activated step). *Let $q \in GL_2(\mathbb{F}_3)$ be nonscalar, with $r_0 = \text{ord}(q)$. Consider any determinant-compatible lifting tower rooted at q . If at some step the exact order first increases from r_0 to $3r_0$, then every determinant-compatible descendant at every later level has its order multiplied by three at each subsequent step. In particular every activated parent has exactly 27 activated children, and the transition $\text{ACT} \rightarrow 27\text{ACT}$ is absorbing.*

Proof. The first order increase supplies a nonzero defect by the exact one-step dichotomy. Lemma 5.11 applies to *every* next lift and recursively to every subsequent lift, showing that the current order triples at each level. The number of determinant-compatible children of a parent is $3^3 = 27$ by the tangent-torsor lemma. Thus all 27 children are activated and the argument repeats indefinitely. $\square$

Remark 5.13 (Transfer cancellation is consistent but not required). *If the current order is $R = 3^a r_0$, $a \geq 1$, then*

$$N_q^{(R)} = 3^a N_q^{(r_0)} = 0 \quad \text{over } \mathbb{F}_3.$$

Thus all children have the same first-order R -defect. The boundary lemma proves more: that common defect is necessarily nonzero after first activation, and gives its exact value at every later level. This removes the nonzero-defect assumption from the earlier conditional activated-step formulation.

Theorem 5.14 (Recursive branching with fixed determinant). *Consider a characteristic-three lifting tower with nonscalar reduction and determinant fixed to $+1$ or -1 at every level. Let PRE denote exact order r_0 , and ACT an order increased by a positive power of three. Then*

$$\boxed{\text{PRE} \longrightarrow 9\text{PRE} + 18\text{ACT}, \quad \text{ACT} \longrightarrow 27\text{ACT}.}$$

Proof. The PRE-Persistence Theorem supplies exactly nine order-preserving and eighteen first-activated children of each PRE parent. Proposition 5.12 shows that each such activated child and all its descendants have strictly increasing powers of three in their exact orders. The tangent-torsor lemma gives exactly 27 children per parent. These are the two claimed transitions. $\square$

6 Consequences and discussion

6.1 Counting consequences

Let P_n and A_n denote the numbers of PRE and ACT descendants after n further lifting levels. Then

$$\begin{pmatrix} P_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 18 & 27 \end{pmatrix} \begin{pmatrix} P_n \\ A_n \end{pmatrix}.$$

After normalization by 27,

$$\begin{pmatrix} 1/3 & 0 \\ 2/3 & 1 \end{pmatrix}$$

is the transition matrix for the corresponding proportions.

Starting from a PRE root, the surviving PRE proportion after d levels is

$$3^{-d}.$$

If each of the 27 determinant-compatible children is selected with equal probability, independently at each level, the first-activation depth therefore has distribution

$$\Pr(D = d) = \frac{2}{3^d}, \quad d = 1, 2, \dots,$$

and

$$\mathbb{E}[D] = \frac{3}{2}.$$

6.2 Discussion and scope

The order defect varies affinely by the cyclic adjoint transfer. This connects a linear-algebraic image and kernel computation to exact counts of order-preserving and order-increasing lifts. In characteristic three, the nonscalar fixed-determinant case yields the branching law of Theorem 5.14; it is not asserted for scalar reductions, arbitrary odd primes, or unrestricted determinant fibers.

For $p > 3$, the nonscalar Jordan transfer vanishes at first order, but the formal perturbative response has the three depths of Theorem 4.6. Infinite depth means exact vanishing of this particular formal matrix response, not an assertion about physical observability. These formulas use classical congruence, Jordan, and modular-transfer ingredients; historical priority of their particular combination is not established here.

6.3 Computational checks

Finite computations were used to test special cases and detect errors, not as premises of the proofs. Checks included the transfer ranks, first-variation identity, characteristic-three lift counts, cubic defect propagation, and Jordan perturbative coefficients at selected small odd primes. Reproducibility requires publication of the exact scripts, test parameters, and outputs; this manuscript does not claim that such a companion archive is currently available.

A Finite verification: scope

The algebraic statements in the main text are proved without relying on enumeration. Auxiliary finite checks covered the characteristic-three nonscalar matrices, determinant-compatible lifts at small levels, and selected Jordan perturbations for $p = 3, 5, 7, 11, 13$. These checks are limited to the tested parameters and are not proofs for all primes or all congruence levels. No particular public code archive, checksum manifest, or permanent identifier is asserted in this version.

Declaration of AI assistance

OpenAI ChatGPT was used for mathematical exploration, drafting, and programming assistance. The author is responsible for the submitted claims, proofs, references, and text. Computational checks and AI assistance do not replace independent mathematical verification.

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